

# APPLICATIONS OF $p$ -DEFICIENCY AND $p$ -LARGENESS

J. O. BUTTON AND A. THILLAISUNDARAM

## INTRODUCTION

At the turn of the twentieth century, William Burnside posed the question: can a finitely generated group which is torsion (meaning that every element has finite order) be infinite? Initially many thought the answer would be negative. Then in 1964, Golod and Shafarevich [10] gave such an example. Afterwards other constructions were obtained by Adjan & Novikov [2], Olshanskiĭ [27], Grigorchuk [12], and Gupta & Sidki [15], to name a few. Very recently Schlage-Puchta [30] gave a remarkably straightforward proof of the existence of such groups by introducing the concept of  $p$ -deficiency, which we review in Section 2.

Now little work has been done on what we call here the related Burnside problem (see [19], Problem 8.52): do there exist infinite finitely *presented* torsion groups? Certainly, no conclusive answer has been established.

We contribute to the related Burnside problem by proving that there do not exist infinite finitely presented torsion groups with  $p$ -deficiency greater than one. This follows from our main result: For a group  $G$  and a prime  $p$ , we say that

- $G$  is *large* (as introduced in [29]) if some (without loss of generality normal) subgroup with finite index admits a non-abelian free quotient;
- $G$  is  *$p$ -large* (as introduced in [21]) if some normal subgroup with index a power of  $p$  admits a non-abelian free quotient.

**Theorem.** *For a finitely presented group  $G$  with  $p$ -deficiency greater than one,  $G$  is  $p$ -large.*

A corollary of this result is that the finitely generated infinite  $p$ -groups constructed by Schlage-Puchta are never finitely presented and do not have property (T).

Note that our main result runs parallel to the famous Baumslag-Pride Theorem [3]: groups with at least two more generators than relators (that is deficiency greater than one) are large. In this proof, the authors show that for a group  $G$  of deficiency greater than one, when  $n$  is sufficiently large,  $G$  has an index  $n$  normal subgroup  $\overline{H}$  that surjects onto  $F_2$ . As a consequence of the proof of this theorem (taking  $n$  to be a suitably large power of  $p$ ), we have that groups of deficiency greater than one are also  $p$ -large for all primes  $p$ . Naturally, our interest lies in groups of deficiency at most one.

Further to the related Burnside problem, we present here various applications of our main result. A significant application is to Coxeter groups where all labels are powers of an odd prime  $p$ , following up on work by Grigorchuk [13] which considered the case  $p = 2$ .

We then consider presentations with  $p$ -deficiency greater than one where the number of relators can be finite or infinite. We compare these with Golod-Shafarevich presentations in [10] and show that for all primes at least 7, any presentation with  $p$ -deficiency greater than one (or even equal to one as long as the usual deficiency is not equal to one) is Golod-Shafarevich. Moreover, although

this is not true in the case of the three exceptional primes, we show that there exists a finite index subgroup which is Golod-Shafarevich.

In the last section we show that groups having presentations with  $p$ -deficiency greater than one are non amenable and discuss related results on having infinite quotients that either have property (T) or which are amenable.

This paper is organised as follows. We begin with a small section that consists of some terminology and notation, as well as relevant facts on  $p$ -groups. Next, we present our main result. Thirdly we give a brief outline on consequences of  $p$ -largeness and then the application to Coxeter groups is illustrated. The subsequent section combines our concept of  $p$ -deficiency with Golod-Shafarevich groups. Lastly, we consider amenability and property (T).

## 1. SOME BASICS

Throughout this paper,  $p$  denotes a prime number,  $\mathbb{N}$  denotes the set of natural numbers  $\{1, 2, \dots\}$ , and  $\mathbb{F}_p$  is the field of  $p$  elements. A group  $G$  is a  $p$ -group if every element of  $G$  has order a  $p$ th power. (Note:  $G$  can be finite or infinite.)

For a group  $G$ , with  $X$  a set of elements in  $G$ ,  $\langle X \rangle$  denotes the subgroup of  $G$  generated by  $X$ , and  $\langle\langle X \rangle\rangle$  denotes the normal closure in  $G$  of the subset  $X$  of  $G$ . We denote the free group on a set  $X$  by  $F(X)$  and the free group of rank  $r$  by  $F_r$ . For  $x, y \in G$ , we write  $[x, y]$  to mean  $xyx^{-1}y^{-1}$ .

A *presentation*  $\langle X | R \rangle$  for a group  $G$  is a set  $X$ , and  $R$  a subset of  $F(X)$ , such that  $G \cong F(X) / \langle\langle R \rangle\rangle$ . The elements  $x \in X$  are called *generators* and  $r \in R$  are called *relators*. A group  $G$  is

- *finitely generated* if there exists some presentation  $\langle X | R \rangle$  of  $G$  with  $|X| < \infty$ ;
- *finitely presented* if there exists some presentation  $\langle X | R \rangle$  of  $G$  with both  $|X|$  and  $|R|$  finite.

For  $H$  a proper subgroup of  $G$ , we say that  $H$  is *subnormal* in  $G$  if there exists a positive integer  $n$  and a normal series in  $G$  such that

$$H = H_n \trianglelefteq H_{n-1} \trianglelefteq \dots \trianglelefteq H_1 \trianglelefteq H_0 = G.$$

For a finite  $p$ -group  $P$ , every finite index subgroup is subnormal. In fact, we can say more:

**Theorem.** [18] *Suppose that  $P$  is a finite  $p$ -group.*

- a) *Every normal subgroup  $N \trianglelefteq P$  may be included into some central series of  $P$  with factors of order  $p$ .*
- b) *Every subgroup  $H \leq P$  may be included into some subnormal series of  $P$  with factors of order  $p$ .*

In an infinite group  $G$  we have that any subnormal subgroup  $H$  of prime power index in  $G$  contains a subgroup  $N$  which is of prime power index and normal in  $G$  (for instance by considering the derived  $p$ -series of  $G$ ).

We note that if  $G$  surjects onto a direct product  $C_p \times \dots \times C_p$ , then  $G$  factors through  $G/G^p$ . In addition, we note that in a finite  $p$ -group  $P$ ,  $P^p P'$  is the Frattini subgroup,  $\Phi(P)$ , of  $P$ . By definition, the Frattini subgroup of an arbitrary group is the intersection of all maximal subgroups. Also the Frattini subgroup is a characteristic subgroup and it consists of all non-generators of the group:  $x \in G$  is a *non-generator* if whenever  $x \in X \subseteq G$  and  $\langle X \rangle = G$  then  $\langle X \setminus \{x\} \rangle = G$ .

We finish this section with a very brief mention of properties (T),  $(\tau)$ , and amenability. A finitely generated group  $G$  has *property* (T) if every isometric action of  $G$  on a Hilbert space has a global fixed point. It has *property*  $(\tau)$  if for some (equivalently any) finite generating set  $S$  for  $G$ , the

set of Cayley graphs  $\text{Cay}(G/N, S)$  form an expander family, where  $N$  varies over all finite index normal subgroups of  $G$ . It is *amenable* if there exists a finitely additive, left invariant, probability measure. All three properties are preserved under quotients, extensions, and subgroups of finite index, whereas amenability is further preserved under arbitrary subgroups. Property (T) implies  $(\tau)$ , but not vice versa. However an amenable group with (T) must be finite, as must a residually finite, amenable group with  $(\tau)$ .

## 2. GROUPS WITH $p$ -DEFICIENCY GREATER THAN ONE

We now recall Schlage-Puchta's beautiful and straightforward construction of infinite finitely generated  $p$ -groups as given in [30].

**Definition.** Let  $G$  be a finitely generated group and  $\langle X|R \rangle$  a presentation for  $G$ . Throughout this paper we always assume that  $X$  is finite, but we allow  $R$  to be finite or infinite, in which case we refer to  $\langle X|R \rangle$  as a finite or infinite presentation respectively. The *deficiency* of  $G$  with presentation  $\langle X|R \rangle$  is

$$\text{def}(G; X, R) = |X| - |R|.$$

If  $|R| = \infty$ , then we define  $\text{def}(G; X, R)$  to be  $-\infty$ .

If we take such a presentation  $G = \langle x_1, \dots, x_d | w_1, w_2, \dots \rangle$  where the  $w_j$  are elements of the free group  $F_d$  then the Reidemeister-Schreier rewriting process (see [23]) will produce a presentation of an index  $i$  subgroup  $H$  of  $G$ , for which there will be  $(d-1)i + 1$  generators. As for relators, each  $w_j$  gives rise to  $i$  relators in this presentation for  $H$ , and these  $i$  relators are conjugate to each other in  $F_d$ . In particular if this presentation is finite, whereupon  $r$  relators for  $G$  results in  $ir$  for  $H$ , we see that  $\text{def} - 1$  is multiplicative, that is for  $H \leq_f G$  with index  $i$  we have a presentation  $H \cong \langle Y|S \rangle$  with  $\text{def}(H; Y, S) - 1 = i(\text{def}(G; X, R) - 1)$ .

Now suppose that  $G$  has a normal subgroup  $H$  of index  $p$  for some prime  $p$ , so that  $H$  can be thought of as the kernel of a homomorphism  $\theta$  from  $G$  onto  $C_p$ . We can assume on changing the presentation that  $\theta(x_1) = \dots = \theta(x_{d-1}) = 0$  but  $x_d$ , which we henceforth rename  $t$ , maps to 1. This is because not every generator is sent to zero, so by reordering them we can assume that  $x_d$  does not. Moreover we can take  $\theta(t) = 1$  without changing the kernel. Next we replace each of the first  $d-1$  generators  $x_i$  with  $x_i t^{-n_i}$  where  $\theta(x_i) = n_i$  and we rewrite the new relators in terms of the new generators. In particular the number of generators and relators is unchanged. The reason for changing the presentation in this way is that the Reidemeister-Schreier process now takes on a particularly simple form. The generators of  $H$  are  $s := t^p$  and  $x_{i,j} := t^j x_i t^{-j}$  for  $1 \leq i \leq d-1$  and  $0 \leq j \leq p-1$ , and the relators are  $t^j w_k t^{-j}$  for  $1 \leq k \leq r$  (or for all  $k \geq 1$  if  $R$  is infinite) and  $0 \leq j \leq p-1$  but written in terms of the above generating set for  $H$ .

Schlage-Puchta's construction rests on two easily verifiable but ingenious observations. The first comes into play when one of the relators for  $G$ , let us say  $w_1$  when written in terms of the new generators for  $G$ , is a  $p$ th power of some other word  $v \in F_d$ . Let our homomorphism from  $G$  to  $C_p$  now have domain  $F_d$  (but we will still name it  $\theta$ ) by composing  $\theta$  with the natural map from  $F_d$  to  $G$ . If  $\theta(v) = 0$  then it is clear that rewriting  $t^i w_1 t^{-i} = (t^i v t^{-i})^p$  in terms of our generators for  $H$  yields the same result as rewriting  $t^i v t^{-i}$  in terms of these generators and raising to the power  $p$ . In particular one  $p$ th power relator in the presentation for  $G$  results in  $p$   $p$ th power relators in that for  $H$ . This cannot be true if  $\theta(v) \neq 0$  because then  $t^i v t^{-i}$  is not in the kernel of  $\theta$  and so cannot be rewritten in terms of these generators. But it can be seen that the  $(t^i v t^{-i})^p$  are conjugate

in  $F_{(d-1)p+1}$  when written in terms of the  $(d-1)p+1$  generators for  $H$ , because on increasing  $i$  by one during rewriting, a cyclic permutation of the previous rewritten word is produced. As conjugates of a given relator can be ignored, in this situation we can say that a  $p$ th power has produced a single relator in the presentation for  $H$ , not  $p$  separate relators. We refer to the above process as *Puchta rewriting*.

The second crucial observation is to quantify the above in such a way that it can work for presentations with infinitely many relators.

**Definition.** [30] For a prime  $p$ , the  $p$ -deficiency of  $G$  with presentation  $\langle X|R \rangle$  is

$$\text{def}_p(G; X, R) = |X| - \sum_{r \in R} p^{-\nu_p(r)},$$

where  $\nu_p(r) = \max \left\{ k \mid \exists w \in F(X), w^{p^k} = r \right\} \geq 0$ .

If  $|R| = \infty$  and the sum does not converge then we define  $\text{def}_p(G; X, R)$  to be  $-\infty$ , although we can certainly have cases where  $|R| = \infty$  but  $\text{def}_p(G; X, R)$  is finite.

Note: it is always the case that  $\text{def}_p(G; X, R) \geq \text{def}(G; X, R)$  and if  $\nu_p(r) = 0$  for all  $r \in R$ , then  $\text{def}_p(G; X, R) = \text{def}(G; X, R)$ .

From the above we see that Schlage-Puchta has proved a multiplicative property for  $\text{def}_p - 1$  similar to that for  $\text{def} - 1$ . That is, if  $H \leq_{p^k} G$  and  $H$  is subnormal in  $G$  then there exists  $Y$  and  $S$  with  $H \cong \langle Y|S \rangle$  such that  $\text{def}_p(H; Y, S) - 1 = p^k (\text{def}_p(G; X, R) - 1)$ .

For ease of notation, we shall in future write  $\text{def}_p(G)$  and  $\text{def}(G)$  instead of  $\text{def}_p(G; X, R)$  and  $\text{def}(G; X, R)$  respectively as we will always have a specific presentation in mind, even though these quantities can vary drastically over all presentations defining  $G$ .

**Definition.** The  $p$ -rank  $d_p(G)$  of a finitely generated group  $G$  is the dimension of the homology group  $H_1(G; \mathbb{F}_p)$ . Equivalently it is the rank (here meaning the number of generators) of  $G/G'G^p$  and can be thought of as the maximum number of copies of  $C_p$  onto which  $G$  surjects:  $G \twoheadrightarrow C_p \times \dots \times C_p$ .

For a presentation  $G = \langle X|R \rangle$  with  $d = |X|$  but where  $R$  is either infinite or finite (with ranges of sums adjusted accordingly), suppose exactly  $s$  of the relators in  $R$  are not powers of  $p$  - that is,  $y \in R$  cannot be expressed as  $v^p$  for some  $v \in F(X)$ . If  $\text{def}_p(G)$  is finite then  $s$  must be finite too. Our presentation for  $G$  thus can be written in the form

$$\langle x_1, \dots, x_d \mid w_1, \dots, w_s, w_{s+1}^{p^{b_{s+1}}}, w_{s+2}^{p^{b_{s+2}}}, \dots \rangle,$$

for  $w_i \in F(X)$  where  $b_{s+1}, b_{s+2}, \dots \geq 1$ . Referring to our definition of  $d_p(G)$ , it is natural to consider  $G/G'G^p$  in our next step. In additive notation, assuming all generators commute, we have

$$\frac{G}{G'G^p} = \langle x_1, \dots, x_d \mid pw = 0 \ \forall w \in F(X), w_1, \dots, w_s \rangle$$

where, for  $1 \leq i \leq s$ , we can express  $w_i = a_{i1}x_1 + \dots + a_{id}x_d$  with  $a_{ij} \in \mathbb{F}_p$  for  $j = 1, \dots, d$ . As  $G$  is finitely generated,  $G/G'G^p$  is a finite abelian group of exponent  $p$  and so can be viewed as a vector space over  $\mathbb{F}_p$ , with  $d$  variables (the generators) and  $s$  not necessarily linearly independent equations (the relators). Thus, as a vector space,  $\dim(G/G'G^p) \geq d - s$ . So we establish the inequality

$$d_p(G) \geq d - s.$$

Now, by definition,  $\text{def}_p(G) = |X| - s - \sum_{i=s+1}^{\infty} p^{-b_i} \leq d - s$ . From the above, we make the key deduction:

$$(2.1) \quad d_p(G) \geq \text{def}_p(G).$$

This has an important consequence.

**Corollary 2.1.** *If there exists a presentation for the finitely generated group  $G$  with  $\text{def}_p(G) \geq 1$  then  $G$  is infinite.*

*Proof.* From (2.1) we see that  $d_p(G) \geq 1$ , so that there exists  $H \trianglelefteq G$  with index  $p$ . But multiplicity of  $p$ -deficiency implies that  $\text{def}_p(H) \geq 1$  so we can iterate indefinitely.  $\square$

Now in order to create finitely generated infinite  $p$ -groups, Schlage-Puchta merely takes a finite generating set  $X$  with  $|X| \geq 2$  and an enumeration  $w_1, w_2, \dots$  of all elements in  $F(X) - \{\text{id}\}$ . On forming the presentation  $G = \langle X | w_1^{p^{n_1}}, w_2^{p^{n_2}}, \dots \rangle$ , we see that  $G$  is clearly a  $p$ -group and is infinite if  $\sum_{i=1}^{\infty} 1/p^{n_i} \leq |X| - 1$  by Corollary 2.1. He also shows that  $G$  is non amenable if the sum is less than  $|X| - 1$  using rank gradient, which will be discussed in Section 6.

The motivation for our main result in this section is the related Burnside problem. As  $p$ -deficiency provides such a natural way of constructing infinite finitely generated torsion groups, an obvious question to ask is whether it can be adapted to produce examples which are finitely presented (see Corollary 2.4 below). However if  $G$  is large then it certainly is not an infinite torsion group.

The following is the goal of this section.

**Theorem 2.2.** (Main Result) *For a finitely presented group  $G$  with  $\text{def}_p(G) > 1$ ,  $G$  is  $p$ -large.*

*Proof.* The following is a vital tool that we will need.

**Theorem 2.3.** ([21] Theorem 1.15) *Let  $G$  be a finitely presented group, and let  $p$  be a prime. Then the following are equivalent.*

- (1)  $G$  is  $p$ -large;
- (2)  $G$  has an abelian  $p$ -series with rapid descent.

We use Schlage-Puchta's multiplicity result to show that the group  $G$  of Theorem 2.2 has an abelian  $p$ -series of rapid descent. Then we apply Theorem 2.3.

**Definition.** [21] An *abelian  $p$ -series* for a group  $G$  is a sequence of finite index subgroups

$$G = G_1 \supseteq G_2 \supseteq \dots$$

such that  $G_i/G_{i+1}$  is an elementary abelian  $p$ -group for each natural number  $i$ .

An abelian  $p$ -series  $\{G_i\}$  has *rapid descent* if

$$\inf_i \frac{d_p(G_i/G_{i+1})}{[G : G_i]} > 0.$$

*Remark.* The equivalence of Theorem 2.3 enables us to conclude that for a  $p$ -large group  $H$  that is subnormal of  $p$ th power index in a supergroup  $G$ , then  $G$  is  $p$ -large. This is apparent from the fact that an abelian  $p$ -series of  $H$  with rapid descent can be extended to an abelian  $p$ -series of  $G$ , still with rapid descent. We also give a direct proof in Lemma 3.4.

Having laid out all the machinery that is required, we construct the required abelian  $p$ -series

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots$$

as follows. Henceforth,  $G \cong \langle X|R \rangle$  is a finitely presented group with  $\text{def}_p(G) > 1$ , as in the hypothesis of Theorem 2.2.

We assume that  $|X| \geq 2$ , as if  $|X| = 1$  then our group  $G$  is cyclic and we cannot possibly have  $\text{def}_p(G) > 1$ . Moreover we know from (2.1) that  $d_p(G) > 1$ , which means a surjection  $G \twoheadrightarrow C_p$  exists. Therefore,  $\exists G_1 \trianglelefteq_p G$ .

We set  $G_0 := G$ . As  $\text{def}_p(G_0) - 1 = \varepsilon > 0$ , Puchta rewriting means that

$$\text{def}_p(G_1) - 1 = p(\text{def}_p(G_0) - 1) = p\varepsilon.$$

If  $p\varepsilon \geq 1$ , then we advance to the next paragraph. Else, proceed down the series to get

$$G_n \trianglelefteq_p G_{n-1} \trianglelefteq_p \dots \trianglelefteq_p G_1 \trianglelefteq_p G_0$$

with  $\text{def}_p(G_n) - 1 = p^n \varepsilon \geq 1$ . The formation of such a series is possible since  $d_p(G_i) \geq \text{def}_p(G_i) > 1$  for each  $G_i$ . We have  $[G : G_n] = p^n$ .

A new abelian  $p$ -series is initiated from  $H := G_n$  (or  $G_1$  if  $p\varepsilon \geq 1$ ). We show instead that  $H$  is  $p$ -large, which implies that  $G$  is  $p$ -large.

Let  $H_1 \trianglelefteq_p H$  (which exists by a similar argument as to  $G_1 \trianglelefteq_p G$ ). Then  $\text{def}_p(H_1) - 1 = p(\text{def}_p(H) - 1) \geq p$ .

So  $d_p(H_1) \geq p + 1$ , but ' $d_p(H_1) \geq p$ ' suffices for our proof, and hence  $H_1 \twoheadrightarrow \underbrace{C_p \times \dots \times C_p}_{p \text{ times}}$ .

Set  $H_2 := \ker(H_1 \twoheadrightarrow (C_p)^p)$ . Then  $H_1/H_2 \cong \underbrace{C_p \times \dots \times C_p}_{p \text{ times}}$ , so  $|H_1/H_2| = p^p$ , and  $d_p(H_1/H_2) = p$ .

We compute that

$$\frac{d_p(H_1/H_2)}{[H : H_1]} = \frac{p}{p} = 1.$$

Following the same line of thought,

$$\text{def}_p(H_2) - 1 = p^p(\text{def}_p(H_1) - 1) \geq p^{p+1} \implies d_p(H_2) \geq p^{p+1}.$$

Similarly,  $H_3 := \ker(H_2 \twoheadrightarrow (C_p)^{p^{p+1}})$ . So  $|H_2/H_3| = p^{p^{p+1}}$ , and hence  $d_p(H_2/H_3) = p^{p+1}$ .

The next fraction gives us the same value.

$$\frac{d_p(H_2/H_3)}{[H : H_2]} = \frac{p^{p+1}}{p^{p+1}} = 1$$

Proceeding in this manner, we have ensured that

$$\inf_i \frac{d_p(H_i/H_{i+1})}{[H : H_i]} = 1 > 0.$$

□

As the reader might have noted, a recursive formula for  $d_p(H_i/H_{i+1})(= [H : H_i])$  is

$$\begin{aligned} d_p(H_i/H_{i+1}) &= p^{d_p(H_{i-1}/H_i)} \cdot d_p(H_{i-1}/H_i) \\ &= p^{[H : H_{i-1}]} \cdot [H : H_{i-1}]. \end{aligned}$$

It is worth pointing out that this brings to light a whole new collection of large groups with finite negative deficiency. For example, on taking one's favourite prime  $p$ , the 2-generator group

$$\langle x, y | w_1^p, \dots, w_{p-1}^p \rangle$$

is  $p$ -large, regardless of which words  $w_1, \dots, w_{p-1} \in F(x, y)$  are chosen.

*Remark.* The proof of Theorem 2.2 works if the given presentation for  $G$  is either finite or infinite. However, in the latter case, suppose the presentation for  $G$  is  $\langle x_1, \dots, x_d | r_1, r_2, \dots \rangle$ . By [25] Corollary 12 of the classic 1937 paper by B. H. Neumann, if  $G$  has some finite presentation then there exists  $l$  with  $G = \langle x_1, \dots, x_d | r_1, \dots, r_l \rangle$ . But as this is a truncation of a presentation with  $p$ -deficiency greater than 1, we obtain a finite presentation for  $G$  with  $p$ -deficiency greater than 1 anyway. This also applies if we have an infinite presentation for  $G$  which has  $p$ -deficiency exactly 1.

We have two corollaries, the first of which is immediate from Theorem 2.2 and this remark.

**Corollary 2.4.** *There do not exist infinite finitely presented torsion groups which, for some prime  $p$ , have a presentation with  $p$ -deficiency greater than 1 or an infinite presentation with  $p$ -deficiency equal to 1.*

In particular the infinite finitely generated  $p$ -groups constructed by Schlage-Puchta are definitely not finitely presented.

Also we can show that these groups do not have property (T).

**Corollary 2.5.** *A group possessing a finite presentation with  $p$ -deficiency greater than 1, or an infinite presentation with  $p$ -deficiency at least 1, does not have property (T).*

*Proof.* We may assume the presentation is infinite. By a result [31] of Shalom, if a finitely generated group has property (T) then only finitely many relations suffice to confirm this. But as in the previous corollary, such a finite presentation defines a large group which cannot have (T). □

*Remark.* Given an infinite presentation  $G = \langle x_1, \dots, x_d | r_1, r_2, \dots \rangle$  as above, define  $G_l = \langle x_1, \dots, x_d | r_1, r_2, \dots, r_l \rangle$ . Suppose that P is a group theoretic property that is preserved by prequotients. We can then ask: if  $G_l$  has P for all  $l$  then does  $G$  have P? We know that this is true for being infinite and not having (T). It also holds for being non abelian, non nilpotent, and having first Betti number or  $p$ -rank at least  $k$ . But by the above it is false for any property implied by largeness but

not held by torsion groups. This includes, for instance, having virtual first Betti number at least  $k$ , or containing a non abelian free group or an element of infinite order.

Two other examples of this phenomenon are in [14] where it is shown that the groups formed by truncating the standard presentation of the Grigorchuk group are all large, as well as [4] Chapter IV Theorem 7 which does the same for the restricted wreath product  $\mathbb{Z} \wr \mathbb{Z}$ . In particular, the above is false for the property of being non amenable, or even being non soluble.

### 3. PROPERTIES OF $p$ -LARGENESS

Largeness was introduced in [29] as an important property of finitely generated groups which is invariant under finite index subgroups and supergroups, as well as prequotients. We now look at the appropriate properties of  $p$ -largeness. First it is clear that  $p$ -largeness is preserved by prequotients because the index and normality of a subgroup are preserved under the inverse image of a homomorphism. However a perfect large group, such as  $A_5 * A_5$ , cannot be  $p$ -large for any  $p$  because of the following. Here we write  $H \leq_f G$  for  $H$  a finite index subgroup of  $G$  and  $N \trianglelefteq_{p^k} G$  when  $N$  is a normal subgroup with index a power of  $p$ .

**Lemma 3.1.** *If the  $p$ -rank  $d_p(G) \leq 1$  then  $G$  is not  $p$ -large.*

*Proof.* Suppose  $P$  is a finite  $p$ -quotient of  $G$ . Then the  $p$ -rank  $d_p(P)$  is also at most 1 so  $P/P'P^p$  is cyclic. But by the Lemma in Section 1 on the Frattini subgroup of a finite  $p$ -group, this means  $P$  is cyclic. However if  $N \trianglelefteq_{p^k} G$  with  $N$  surjecting onto the free group  $F_2$  of rank 2 then  $N'N^p$  is characteristic in  $N$ , hence normal in  $G$ , and  $N/N'N^p$  surjects onto  $C_p \times C_p$ . Thus  $G/N'N^p$  is a non cyclic finite  $p$ -group.  $\square$

Also we cannot remove normality from the definition of  $p$ -large.

#### Example 3.2.

If  $G$  has a non (sub)normal subgroup of index a power of  $p$  that surjects onto  $F_2$  then  $G$  need not be  $p$ -large. For instance let  $G = \mathbb{Z} * C_2$ . As this has  $p$ -rank 1 unless  $p = 2$ ,  $G$  cannot be  $p$ -large for primes  $p \geq 3$  by Lemma 3.1, although it is 2-large and hence large. However  $G$  has a non normal subgroup of index 3 which is  $F_2 * C_2$ .

**Lemma 3.3.** *If  $G$  is  $p$ -large and  $H \leq_f G$  then  $H$  is  $p$ -large.*

*Proof.* Suppose  $N \trianglelefteq_{p^k} G$  with  $N$  surjecting onto  $F_2$ . Then  $H \cap N$  is normal and of finite index in both  $N$  and  $H$ , so  $H \cap N$  surjects onto a non abelian free group by restricting the homomorphism from  $N$ . Moreover the index  $[H : H \cap N] = [HN : N]$  and as  $N$  normal implies that  $HN$  is a subgroup of  $G$ , we have  $[HN : N]$  divides  $p^k$ .  $\square$

Now let us consider supergroups of finite index, so suppose that  $H$  is  $p$ -large and  $H \leq_f G$ . Then  $G$  need not be  $p$ -large, even if the index is  $p$ , as in Example 3.2. Also it is not true even if  $H$  is normal in  $G$  and the index is coprime to  $p$ , or divides  $p$ , for instance  $C_2 * C_2 * C_2$  is not  $p$ -large for  $p > 2$  but  $G$  has free non abelian normal subgroups of index  $2n$  for all  $n \in \mathbb{N}$ . But the one case where we can transfer  $p$ -largeness to a finite index supergroup is when  $H$  is normal in  $G$  and has index a power of  $p$ . Indeed subnormality works here:

**Lemma 3.4.** *If  $N$  is  $p$ -large and is a subnormal subgroup of  $G$  with index  $p^k$  then  $G$  is  $p$ -large.*



*Proof.* There exists  $M \trianglelefteq_{p^l} N$  with  $M$  surjecting onto  $F_2$ . Although  $M$  need not be normal in  $G$ , it is subnormal with index  $p^{k+l}$ . Therefore we have a subgroup  $L \trianglelefteq_{p^m} G$  with  $L \leq M$ , as mentioned in Section 1, and  $L$  surjects onto  $F_2$  as well.  $\square$

An important consequence of a finitely generated group being large is that it has many finite quotients. We have a variant on this which is that if  $G$  is  $p$ -large then it has many  $p$ -groups as quotients, where we include both finite and infinite  $p$ -groups. Theorem 4 in [24] states that if  $G$  is large with  $N \trianglelefteq_f G$  mapping onto  $F_2$  and  $p$  is any prime then  $G$  possesses uncountably many residually finite torsion quotients (here we always count up to isomorphism) such that in each quotient the image of  $N$  is a  $p$ -group. Moreover Corollary 1 of this paper shows that if  $G$  is  $p$ -large then  $G$  possesses uncountably many residually finite quotients that are  $p$ -groups.

We can give here a shorter proof of these facts provided we assume a priori that for each prime  $p$  there exist uncountably many finitely generated residually finite  $p$ -groups which are just infinite (for instance, see [11]). To do this we will adapt Proposition 4.5 in [7] which shows that if  $H \leq_f G$  where  $H$  maps onto an infinite group with property (T) then so does  $G$ . In fact property (T) is not used directly in the proof, thus it can be replaced by other suitable properties. Moreover a very similar approach was earlier used by H. N. Neumann in [26] Section 2 and was also known to J. S. Wilson in [32].

**Proposition 3.5.** *Suppose that  $P$  is a property of finitely generated groups which is preserved by quotients, finite index subgroups and supergroups, and finite direct products. If  $G$  is finitely generated and has  $H \leq_f G$  such that  $H$  maps onto an infinite group with  $P$  then so does  $G$ .*

*Proof.* Take  $K \trianglelefteq_f G$  with  $K \leq H$ . Then  $K$  also has an infinite quotient with  $P$  by restriction. As  $P$  is preserved by quotients, we can assume that there is  $L \trianglelefteq K$  where  $K/L$  is just infinite and has  $P$ . Although we need not have  $L \trianglelefteq G$ , the normaliser of  $L$  in  $G$  contains  $K$  and so it has finite index in  $G$ . Consequently we can assume that  $L_1, \dots, L_n$  are all of the conjugates of  $L$  in  $G$ . On putting  $N = L_1 \cap \dots \cap L_n$  we have  $N \trianglelefteq G$  and a natural injective homomorphism

$$\pi : K/N \rightarrow \Gamma = K/L_1 \times \dots \times K/L_n$$

which is surjective under each projection onto a factor, so that  $K/N$  is a subdirect product of  $K/L \cong K/L_i$ . Moreover if we let  $\Pi = \pi(K/N) \cong K/N$  and  $\Pi_i = \Pi \cap (K/L_i)$ , we have that  $\Pi_i$  is normal in  $K/L_i$ .

We can assume that  $\Pi_i$  has finite index in the just infinite group  $K/L_i$ , because otherwise  $\Pi_i$  is trivial in which case the map from  $\Gamma$  which forgets the  $i$ th component is injective when restricted to  $\Pi$ . Now  $\Pi$  contains  $\Pi_1 \times \dots \times \Pi_n$  which has finite index in  $\Gamma$ , so  $\Gamma$  having  $P$  and  $\Pi \leq_f \Gamma$  means that  $\Pi$  does too. Thus  $G/N$  has the finite index subgroup  $K/N$  with  $P$ , so  $G/N$  has  $P$  and is infinite.  $\square$

Thus if a finitely generated group  $G$  has  $H \leq_f G$  where  $H$  has an infinite quotient with (T), or  $(\tau)$ , or which is amenable then so does  $G$ . In fact the argument can be simplified in the amenable case: if the property  $P$  in Proposition 3.5 is preserved by all subgroups, finite index supergroups and finite direct products, but not necessarily by quotients, then we do not need  $K/L$  to be just infinite in the proof because if  $K/L$  has  $P$  then so will  $K/N$ , as it is a subgroup of  $\Gamma$ .

In the original argument by H. M. Neumann, the specific property considered is that of containing a given infinite simple group  $S$ . However the features required of this property for the proof to work are that it is preserved by finite index subgroups, all supergroups, and (in place of the just

infinite quotient) a maximal condition which says if  $K$  has a quotient  $K/L_0$  with this property then there exists  $L_0 \leq L \trianglelefteq K$  such that  $K/L$  and any non trivial normal subgroup of  $K/L$  also have this property. We then proceed exactly as in the first paragraph of Proposition 3.5 and conclude that either  $\Pi_i$  has our property, so  $\Pi$  and hence  $G/N$  does, or  $\Pi_i$  is trivial but this cannot happen for all  $i$ .

**Corollary 3.6.** *If  $G$  is large with  $N \trianglelefteq_f G$  mapping onto  $F_2$  and  $p$  is any prime then  $G$  possesses uncountably many residually finite torsion quotients such that in each quotient the image of  $N$  is a  $p$ -group. Moreover if  $G$  is  $p$ -large then  $G$  possesses uncountably many residually finite quotients that are  $p$ -groups.*

*Proof.* As there exist uncountably many finitely generated residually finite  $p$ -groups which are just infinite, there must exist an integer  $n$  such that uncountably many of these are generated by  $n$  elements and so are quotients of  $F_n$ . If  $K \trianglelefteq_f G$  and  $K$  maps onto any non-abelian free group  $F$  then we can assume  $K$  maps onto  $F_n$  by taking  $C$  characteristic and of finite index in  $K$  such that  $C$  surjects onto a free group of rank at least  $n$ . For instance we can take  $C = K'K^q$  where  $q$  is a power of  $p$ , which is characteristic in  $K$  and will map onto  $F'F^q$ . By taking a suitably big power,  $F'F^q$  will have arbitrarily high index in  $F$  and hence arbitrarily high rank. We can then replace  $C$  by  $K$ , ensuring that  $K$  has uncountably many residually finite just infinite  $p$ -groups.

Now if  $P$  is the property of being a torsion group then the conditions for Proposition 3.5 are satisfied. On following the proof, we see that if  $K$  surjects onto the just infinite  $p$ -group  $K/L$  then  $G$  surjects onto the infinite torsion group  $G/N$ , with  $K/N$  a subdirect product of  $K/L$ , thus  $K/N$  is an infinite  $p$ -group.

Being residually finite does not satisfy the appropriate properties but, as noted in [32], if  $K/L$  is residually finite then so is  $G/N$ : On taking  $x \notin N$  (but which we can assume is in  $K$ ) then, as  $N = \bigcap gLg^{-1}$  over all  $g \in G$ , there exists  $g \in G$  with  $g^{-1}xg \notin L$ . So we can take a finite index normal subgroup  $M/L$  of  $K/L$  that misses  $g^{-1}xgL$ . Thus  $x$  is not contained in the finite index subgroup  $gMg^{-1}$  of  $G$  but  $N$  is.

So we have an uncountable set  $\{L_i : i \in I\}$  with  $L_i \trianglelefteq K$  where, by using the above construction, each  $L_i$  gives rise to  $N_i \trianglelefteq G$  such that  $G/N_i$  is an infinite residually finite torsion group. Now if only countably many  $N_i$  (up to isomorphism of  $G/N_i$ ) occur then we must have a particular  $N_{i_0}$  which is obtained from uncountably many  $L_i$ . But then  $K/N_{i_0}$  is a finite index subgroup of  $(K/L_i)^{n_i}$  for each of these  $L_i$  and some  $n_i \in \mathbb{N}$ . Now there are uncountably many isomorphism classes when we vary over all  $(K/L_i)^{n_i}$  because each one contains the finitely generated subgroup  $K/L_i$  and a finitely generated group has only countably many finitely generated subgroups. But the finitely generated group  $K/N_{i_0}$  can only sit with finite index in countably many supergroups (up to isomorphism).

Finally if  $K \trianglelefteq_{p^k} G$  then we can take  $C$  as before, so that on replacing  $C$  with  $K$  and using the above, we now have  $K \trianglelefteq_{p^l} G$  with  $K/N$  and  $G/K$  both  $p$ -groups, so  $G/N$  is too.  $\square$

#### 4. APPLICATION TO COXETER GROUPS

In this section, we extend a result of Grigorchuk [13] to odd primes.

**Definition.** [13] A *Coxeter group* is a group with presentation,

$$C = \langle x_1, \dots, x_n | x_1^2, \dots, x_n^2, (x_i x_j)^{m_{ij}}, 1 \leq i < j \leq n \rangle,$$

where  $m_{ij}$  is an integer at least 2, or  $\infty$  (the case  $m_{ij} = \infty$  means that the relation  $(x_i x_j)^{m_{ij}}$  is not present).

We also note a dichotomy of Coxeter groups, which is listed under Theorem 14.1.2 of [6].

**Theorem 4.1.** [6] *A Coxeter group is either virtually abelian or large.*

Grigorchuk's result is the following.

**Theorem 4.2.** [13] *Every Coxeter group, that is not virtually abelian and for which all labels  $m_{ij}$  (referring to the presentation above) are powers of 2 or infinity, surjects onto uncountably many infinite residually finite 2-groups.*

In order to extend Grigorchuk's result to odd primes, it is necessary to alter our focus to subgroups of Coxeter groups. This is because, for  $n \in \mathbb{N}$  and  $p$  an odd prime, Coxeter groups of the form

$$G = \langle x_1, \dots, x_n | x_1^2, \dots, x_n^2, (x_i x_j)^{m_{ij}}, 1 \leq i < j \leq n, \text{ \& } m_{ij} \text{ is a } p\text{th power (including } \infty) \rangle,$$

do not admit surjections onto  $p$ -groups. To see why this is the case, we note that all generators of  $G$  are of order two, and so they must be mapped to the identity in a  $p$ -group (since  $p$  is an *odd* prime, there are no elements of order two in a  $p$ -group).

Therefore we consider instead the index two normal subgroup  $P$  of  $G$ , which has presentation,

$$P = \langle a_1, \dots, a_{n-1} | a_1^{m_{12}}, \dots, a_{n-1}^{m_{1n}}, (a_{i-1} a_{j-1}^{-1})^{m_{ij}} \rangle$$

where  $2 \leq i < j \leq n$  and  $m_{rs}$ ,  $1 \leq r < s \leq n$ , is as given in  $G$ . This normal subgroup  $P$  has the geometric interpretation as the “orientation-preserving” subgroup of  $G$ . The above presentation for  $P$  is obtained via a straightforward application of the Reidemeister-Schreier rewriting process (details in Subsection 4.1).

Before we state our result, we give a name to these “orientation-preserving” subgroups of Coxeter groups where the labels are all powers of  $p$ .

**Definition.** For  $p$  an odd prime and  $n \geq 2$ , suppose that  $C$  is a Coxeter group with all labels  $m_{ij}$  a power of  $p$  (or  $\infty$ ). A  $p$ -Coxeter subgroup is the index 2 subgroup of  $C$  given by the kernel of  $\theta : C \rightarrow C_2$  where  $C(x_i) = 1$  for all  $i$ . It has the form

$$\langle a_1, \dots, a_{n-1} | a_1^{m_{12}}, \dots, a_{n-1}^{m_{1n}}, (a_{i-1} a_{j-1}^{-1})^{m_{ij}} \rangle$$

where  $2 \leq i < j \leq n$  and  $m_{rs}$ ,  $1 \leq r < s \leq n$  are  $p$ th powers (or infinity).

Here is our result of this section, and we embark on its proof in what follows.

**Theorem 4.3.** *Every  $p$ -Coxeter subgroup that is not virtually abelian surjects onto uncountably many infinite residually finite  $p$ -groups.*

It is insufficient to establish that the relevant  $p$ -Coxeter subgroups are *large* for the result to hold, but  $p$ -largeness is enough by Corollary 3.6. It so happens that all  $p$ -Coxeter subgroups that are not virtually abelian, and hence large, do satisfy the theorem. The virtually abelian

$p$ -Coxeter subgroups are all the cyclic ones and  $\langle a_1, a_2 | a_1^3, a_2^3, (a_1 a_2^{-1})^3 \rangle$ , as we shall see from the proof of Theorem 4.3. For this last group,  $\langle a_1, a_2 | a_1^3, a_2^3, (a_1 a_2^{-1})^3 \rangle$ , which is isomorphic to  $T = \langle a_1, a_2 | a_1^3, a_2^3, (a_1 a_2)^3 \rangle$ , there is a useful geometric interpretation. Its supergroup

$$G = \langle x_1, x_2, x_3 | x_1^2, x_2^2, x_3^2, (x_1 x_2)^3, (x_1 x_3)^3, (x_2 x_3)^3 \rangle,$$

can be viewed as the group generated by reflections in the sides of an equilateral triangle. The group  $G$  has  $\mathbb{Z} \times \mathbb{Z}$  as an index six subgroup (the subgroup of translations), and likewise  $T$  has  $\mathbb{Z} \times \mathbb{Z}$  as an index three subgroup.

*Proof.* For convenience, we will refer to the statement “there exists uncountably many surjections to infinite residually finite  $p$ -groups” as  $(\dagger)$ , and we denote

$$S_n(p) := \langle a_1, \dots, a_{n-1} | a_1^p, \dots, a_{n-1}^p, (a_{i-1} a_{j-1}^{-1})^p, 2 \leq i < j \leq n \rangle.$$

We have  $n \geq 3$  or else  $S_n(p)$  is cyclic. For  $p > 3$ , the  $p$ -deficiency of the following  $p$ -Coxeter subgroup

$$S_3(p) = \langle a_1, a_2 | a_1^p, a_2^p, (a_1 a_2^{-1})^p \rangle$$

is  $2 - \frac{3}{p} > 1$ . So  $S_3(p)$  is  $p$ -large by Theorem 2.2 and so it satisfies  $(\dagger)$ .

Next, we note the following family of surjections, as was done similarly in Grigorchuk’s paper [13]. Here the indices are understood to range as in the definition of  $p$ -Coxeter subgroups.

$$(4.1) \quad \langle a_1, \dots, a_{n-1} | a_1^{p^{k_{12}}}, \dots, a_{n-1}^{p^{k_{1n}}}, (a_{i-1} a_{j-1}^{-1})^{p^{k_{ij}}} \rangle \twoheadrightarrow \langle a_1, \dots, a_{n-1} | a_1^p, \dots, a_{n-1}^p, (a_{i-1} a_{j-1}^{-1})^p \rangle$$

where  $k_{rs} \in \mathbb{N}$  for  $1 \leq r < s \leq n$ ; and

$$(4.2) \quad \langle a_1, \dots, a_n | a_1^p, \dots, a_n^p, (a_{i-1} a_{j-1}^{-1})^p \rangle \twoheadrightarrow \langle a_1, \dots, a_{n-1} | a_1^p, \dots, a_{n-1}^p, (a_{i-1} a_{j-1}^{-1})^p \rangle.$$

Let  $p > 3$ . Since  $S_{n+1}(p) \twoheadrightarrow S_n(p)$  by the surjection (4.2) and  $S_3(p)$  satisfies  $(\dagger)$ , by composition of surjections of type (4.2), we deduce that  $S_n(p)$  satisfies  $(\dagger)$  for all  $n \geq 3$ . From the family of surjections (4.1), we conclude that all  $p$ -Coxeter subgroups with  $p > 3$  satisfy  $(\dagger)$ .

For  $n = 3$ , all such 3-Coxeter subgroups, apart from  $\langle a_1, a_2 | a_1^3, a_2^3, (a_1 a_2^{-1})^3 \rangle$ , have 3-deficiency bigger than 1. For the last remaining collection of 3-Coxeter subgroups with  $n > 3$ , we have proven that  $S_4(3)$  is 3-large with the aid of MAGMA (details in Subsection 4.2). So as before,  $(\dagger)$  holds for all 3-Coxeter subgroups with  $n > 3$ .  $\square$

#### 4.1. Reidemeister-Schreier rewriting for $p$ -Coxeter subgroups.

For every Coxeter group

$$C = \langle x_1, \dots, x_n | x_1^2, \dots, x_n^2, (x_i x_j)^{m_{ij}} \rangle$$

where  $1 \leq i < j \leq n$ ,  $m_{ij} \in \{2, 3, \dots\} \cup \{\infty\}$ , there exists an index two normal subgroup  $P$  which has presentation

$$P = \langle a_1, \dots, a_{n-1} | a_1^{m_{12}}, \dots, a_{n-1}^{m_{1n}}, (a_{i-1} a_{j-1}^{-1})^{m_{ij}} \rangle$$

where  $2 \leq i < j \leq n$  and  $m_{rs}$ ,  $1 \leq r < s \leq n$ , is as given in  $C$ .

The above presentation for  $P$  is obtained via the Reidemeister-Schreier rewriting method: we define  $P$  as the kernel of the homomorphism  $G \twoheadrightarrow \{0, 1\}$  given by considering the parity of word lengths. Our Schreier transversal is taken to be the set  $T = \{e, x_1\}$ .

With this transversal, we obtain generators for  $P$  of the form  $tx(\overline{tx})^{-1}$ , where  $t \in T$  and  $x \in \{x_1, \dots, x_n\}$  and  $\overline{tx}$  is the coset representative of the element  $tx$ .

For  $t = e$ :

$$\begin{aligned} x = x_1 : \quad & x_1 x_1^{-1} = e \\ x = x_2 : \quad & x_2 x_1^{-1} = x_2 x_1 =: a_1 \text{ (note: } x_1 \text{ is of order 2)} \\ & \vdots \\ x = x_n : \quad & x_n x_1^{-1} = x_n x_1 =: a_{n-1} \end{aligned}$$

For  $t = x_1$ :

$$\begin{aligned} x = x_1 : \quad & x_1^2 = e \\ x = x_2 : \quad & x_1 x_2 = a_1^{-1} \text{ (this equality is obtained using the relators } x_1^2, \dots, x_n^2) \\ & \vdots \\ x = x_n : \quad & x_1 x_n = a_{n-1}^{-1} \end{aligned}$$

Now the relators of  $P$  are

$$(x_1 x_2)^{m_{12}} = a_1^{m_{12}}, \dots, (x_1 x_n)^{m_{1n}} = a_{n-1}^{m_{1n}}$$

and

$$(x_i x_j)^{m_{ij}} = (x_i x_1 x_1 x_j)^{m_{ij}} = (a_{i-1} a_{j-1}^{-1})^{m_{ij}} \text{ for } 2 \leq i < j \leq n.$$

The other relators of  $G$ , that is  $x_1^2, \dots, x_n^2$ , were absorbed above during our construction of the generating set.

#### 4.2. The use of MAGMA.

*Claim.*  $P_0 := S_4(3) = \langle a_1, a_2, a_3 \mid a_1^3, a_2^3, a_3^3, (a_1 a_2)^3, (a_1 a_3)^3, (a_2^{-1} a_3)^3 \rangle$  is 3-large. (Note: we have replaced generators  $a_2, a_3$  as in the definition of a  $p$ -Coxeter subgroup, by  $a_2^{-1}, a_3^{-1}$  for convenience).

*Proof.* Using MAGMA's LowIndexNormalSubgroups function, we considered the following index three normal subgroup of  $P_0$ :

$$P_1 = \langle x, y, z, w \mid y^3, w^3, [x, z], (w^{-1} y^{-1})^3, w^{-1} y^{-1} z^{-1} w x^{-1} y z x \rangle,$$

which was thirteenth on the list of fourteen normal subgroups with index at most three in  $P_0$ . The above presentation for  $P_1$  was obtained using MAGMA's Simplify function.

Using MAGMA's LowIndexNormalSubgroups function a further time, we applied MAGMA's Simplify command to the first index three normal subgroup of  $P_1$  on MAGMA's list:

$$\begin{aligned} P_2 = \langle a, b, c, d, e, f \mid & b^3, e^3, [c, a^{-1}], (b^{-1} e^{-1})^3, \\ & f e^{-1} a^{-1} b c a f^{-1} a^{-1} c^{-1} b^{-1} a e, f^{-1} a^{-1} b c a f c^{-1} d^{-1} b^{-1} e^{-1} d e, a f b d c b d^{-1} b^{-1} f^{-1} a^{-1} c^{-1} b^{-1} \rangle. \end{aligned}$$

Then we formed the quotient

$$P_2 / \langle\langle a, b, c, e \rangle\rangle$$

which is isomorphic to  $\langle d, f \rangle \cong F_2$ . Hence  $P_2$  is 3-large by definition, and since  $P_2$  is subnormal in  $P_0$  of index  $3^2$ , we have proved that  $P_0 = S_4(3)$  is 3-large, as required.  $\square$

#### 4.3. More $p$ -large groups.

At the end of [13], Grigorchuk mentions the  $p$ -groups ( $p \geq 3$  is a prime) of Gupta-Sidki [15] that are 2-generated, residually finite, branch, and just-infinite. Their generators  $x, y$  satisfy the relators  $x^p, y^p, (x^i y^j)^{p^2}$ ,  $1 \leq i, j \leq p-1$ , as well as infinitely others. We look at the group

$$G = \langle x, y | x^p, y^p, (x^i y^j)^{p^2} \rangle$$

which surject onto these Gupta-Sidki  $p$ -groups.

Note that, as

$$(x^i y^j)^{-1} = y^{-j} x^{-i} = y^{p-j} x^{p-i} = y^{p-j} (x^{p-i} y^{p-j}) y^{-(p-j)},$$

we only need the relators  $x^p, y^p$ , and  $(x^i y^j)^{p^2}$  with  $1 \leq i \leq (p-1)/2$ ,  $1 \leq j \leq p-1$ , so  $\text{def}_p(G) = 2 - \frac{2}{p} - \frac{(p-1)^2}{2p^2}$  which is greater than 1 for  $p \geq 3$ . Thus, our groups are  $p$ -large.

The paper then puts forward the following question:

**Problem** Let  $p \geq 5$  be prime. Does there exist an infinite, residually finite  $p$ -group generated by two elements  $x, y$  subject to the relations  $x^p = y^p = 1$  and  $(x^i y^j)^p = 1$  for  $1 \leq i, j \leq p-1$ ?

By Corollary 3.6 there are uncountably many such examples if the group  $G(p)$  given by this finite presentation is  $p$ -large. Unfortunately the  $p$ -deficiency of all these presentations is now well below 1. However we can confirm that we do have  $p$ -largeness for  $G(5)$  and  $G(7)$ . This was done by computation and we now give brief details: the paper [5] describes an algorithm in MAGMA which aims to show that, on input of a finite presentation, the group  $G$  so defined is large. The main theoretical tool that is used comes from [17]. This states that if  $G$  has a homomorphism  $\chi$  onto  $\mathbb{Z}$  with kernel  $K$  and there exists a prime  $p$  such that the mod  $p$  Alexander polynomial  $\Delta(t)$  (with respect to  $\chi$ ) is zero then the finite index normal subgroup  $KG^n$  of  $G$  maps onto the free product  $C_p * C_p * C_p$  for sufficiently large  $n$ . Now  $KG^n$  is certainly  $p$ -large because the free product is too so, on taking  $n$  to be a power of  $p$ , we have that  $G$  is  $p$ -large by Lemma 3.4.

However  $G(p)$  has no homomorphisms onto  $\mathbb{Z}$ , but we are done if we can find (sub)normal subgroups of  $p$  power index that do and which satisfy the above condition on some mod  $p$  Alexander polynomial.

For  $G(5)$  we found that a subnormal subgroup of index 25 with abelianisation  $(C_5)^2 \times \mathbb{Z}^4$  satisfies this condition. The computation was instant. For  $G(7)$  the same was true for index 49 and the abelianisation was  $(C_7)^{10} \times \mathbb{Z}^6$ . Here the computation took a few hours to find subgroups of this index, but the Alexander polynomial computation was instant.

## 5. PUCHTA GROUPS VERSUS GOLOD-SHAFAREVICH GROUPS

For convenience, we define the following.

**Definition.** We call a finitely generated group  $G$  a *Puchta group* if  $G$  has a presentation  $\langle X|R \rangle$ , with  $|X| < \infty$ , such that there exists a prime  $p$  with  $\text{def}_p(G; X, R) > 1$ , and we call  $\langle X|R \rangle$  a *Puchta presentation*.

Note that  $G$  will always have other presentations which are not Puchta.

In this section, we relate Puchta groups to Golod-Shafarevich groups. Before we define a Golod-Shafarevich group, we first note that the Zassenhaus  $p$ -filtration  $\{G_n\}$  is given by  $G_1 = G$ ,  $G_2 = [G, G]G^p$ ,  $G_3 = [G_2, G]G_{r_{3/p}^p}$ ,  $\dots$ ,  $G_i = [G_{i-1}, G]G_{r_{i/p}^p}$ .

*Remark.* The Zassenhaus  $p$ -filtration (and hence the lower central  $p$ -series) intersects in the identity for a free group  $F(X)$ .

**Definition.** (a) Consider a group presentation  $\langle X|R \rangle$ , where  $X$  is finite but  $R = \{r_1, r_2, \dots\}$  could be finite or infinite. For the free group  $F(X)$  and any non identity  $w \in F(X)$ , the *degree*  $\deg(w)$  is the maximum  $i$  such that  $w \in F(X)_i$ . The presentation  $\langle X|R \rangle$  is said to satisfy the *Golod-Shafarevich condition* with respect to  $p$  if there exists a real number  $t \in (0, 1)$  such that

$$1 - H_X(t) + H_R(t) < 0 \text{ where } H_X(t) = |X|t \text{ and } H_R(t) = \sum_{i=1}^{\infty} t^{\deg(r_i)}.$$

We call  $F(t) = 1 - H_X(t) + H_R(t)$  the *Golod-Shafarevich function* (or *polynomial* if  $R$  is finite) for this presentation.

(b) A group  $\Gamma$  is called *Golod-Shafarevich*, or *GS* for short, if it has a presentation satisfying the Golod-Shafarevich condition. It is shown in [9] that  $\Gamma$  is infinite, and a major strengthening of this due to Zelmanov in [33] is that the pro- $p$  completion  $\hat{\Gamma}_p$  contains a non abelian free pro- $p$  group.

We also have a stronger condition: for  $G$  finitely presented with  $d_p(G) \geq 2$ ,  $G$  is a *strongly Golod-Shafarevich group* if there exists a finite presentation  $\langle X|R \rangle$  for  $G$  such that

$$(5.1) \quad \text{def}(G; X, R) + \frac{(d_p(G))^2}{4} - d_p(G) > 0.$$

That this implies the GS condition can be seen by assuming that all relators of degree at least two are equal to two, then examining the associated quadratic function.

*Remark.* An important property of degree with respect to the prime  $p$  that will be used is  $\deg(w^p) = p \deg(w)$  for any non identity  $w \in F(X)$ .

**Question.** Are Puchta groups Golod-Shafarevich groups?

Answer: not always. Consider, for example,  $p = 2$  and the presentation  $P = \langle x, y|x^2 \rangle$ . Then  $\text{def}_2(P) = 2 - 1/2 > 1$ . But  $F(t) = 1 - H_X(t) + H_R(t)$  is  $(1 - t)^2$ . Another example for  $p = 2$  which does not even touch the  $t$ -axis is  $P = \langle x, y, z|x^2, y^2, z^2 \rangle$  with  $\text{def}_2(P) = 3/2$  again and

$F(t) = 1 - 3t + 3t^2 = (1 - 3t/2)^2 + 3t^2/4$ . Moreover we can show that  $P$  cannot be GS with respect to any presentation  $\langle X|R \rangle$ . Suppose so then we must have  $|R| \geq |X|$  because  $P$  has finite abelianisation. But if  $r$  is the number of relators of degree 1 in  $R$  then  $d_2(P) = 3$  implies that  $r \geq |X| - 3$ . In particular we can assume that  $F(t) = 1 - 3t + t^{i_1} + \dots + t^{i_n}$  where  $n = |R| - |X| + 3 \geq 3$ . We claim that at least three of these  $n$  remaining relators have degree at most 2. This is because the quotient of  $F(X)$  by the third subgroup  $F(X)_3$  in the Zassenhaus 2-filtration is a nilpotent group of class 2 with abelianisation  $(C_4)^{|X|}$ . However  $F(X)/F(X)_3$  surjects onto the equivalent quotient  $P/P_3$ , and this has abelianisation  $(C_2)^3$ . Consequently at least  $|X|$  non trivial relations are needed in  $F(X)/F(X)_3$  to get  $G/G_3$  as a quotient, but relators of degree 3 or more in  $F(X)$  become trivial relators in  $F(X)/F(X)_3$ .

We can also turn  $P$  into a torsion group by adding to the set of relators,  $R$ , the following:

$$w^{2^{N(w)}} \text{ for } w \in F(x, y, z) \text{ and } N(w) \gg 0,$$

maintaining the condition  $\text{def}_2(P) > 1$ . Certainly, this group is still not a Golod-Shafarevich group with respect to any presentation as the previous argument works here too.

We now find all examples of the above form.

**Lemma 5.1.** *Let  $p$  be any prime and let  $G = \langle x_1, \dots, x_k | w_1^p, \dots, w_l^p \rangle$  be a Puchta presentation such that  $w_1, \dots, w_l$  all have degree 1 in  $F_k$ . Then this presentation is GS unless  $p = 2$  and  $(k, l) = (2, 1), (3, 3), (4, 4), (4, 5), (6, 6)$ ;  $p = 3$  and  $(k, l) = (2, 2), (3, 4), (3, 5)$ ; or  $p = 5$ ,  $k = 2$  and  $l$  equal to 3 or 4.*

*Proof.* In order for the presentation to be Puchta, we need  $p(k-1) > l$  and for it to be GS we must find out whether  $F(t) = 1 - kt + lt^p$  travels below the  $t$ -axis when  $t \in (0, 1)$ . We have that the only point  $u > 0$  where  $F'(u) = 0$  satisfies  $pl u^{p-1} = k$ . We can assume here that  $u < 1$ : otherwise  $k \geq pl$ , and if  $k > l + 1$  then  $F(1) < 0$  anyway, so we are left with  $k \leq l + 1$  giving  $l + 1 \geq pl \geq 2l$ , giving just  $l = 1$ ,  $k = p = 2$  and  $F(t) = (1 - t)^2$ , or  $l = 0$  and  $k = 1$  but this has  $p$ -deficiency 1 for all  $p$ .

Otherwise we look at the sign of  $F(u) = 1 - ku + ku/p$ . We have  $F(u) \geq 0$  exactly when  $lp^p \geq k^p(p-1)^{p-1}$ . Thus in order for the presentation to be Puchta but not GS, we require  $p^{p+1}(k-1) > k^p(p-1)^{p-1}$ . Now for  $n$  and  $k$  integers at least 2, we have by induction on  $n$  that  $k^n/(k-1) \geq 2^n$ . Thus we need primes  $p$  such that  $p^{p+1} > 2^p(p-1)^{p-1}$ . Again we show by induction that  $2^n(n-1)^{n-1} \geq n^{n+1}$  for  $n \geq 7$ , with the base case certainly true. The inductive step is left as an exercise for the reader, with the hint that  $(n+1)^3 \leq 2n^3$  holds for such  $n$ . Thus we have no examples unless  $p = 2, 3$  or  $5$ . Indeed for  $p \geq 7$  we do not even have examples where the presentation has  $p$ -deficiency equal to 1 unless the deficiency is 1 because we do not even have equality in the above.

As for the three exceptional primes, for  $p = 2$  we need  $k^2 - 8k + 8 < 0$  leaving only  $k = 2, 3, 4, 5, 6$ . Now for each value of  $k$  we see which integers  $l$  satisfy both  $p(k-1) > l$  and  $lp^p \geq k^p(p-1)^{p-1}$ , giving us the answer above. For  $p = 3$  we need  $4k^3 - 81k + 81 < 0$  which is increasing for  $k \geq 3\sqrt{3}/2$  so we need only check  $k = 2, 3$  for the 2 equations involving  $l$ . Finally  $p = 5$  gives rise to  $4^4k^5 - 5^6k + 5^6 < 0$  and the left hand side is increasing for at least  $k \geq 2$ , so we only check  $k = 2$  and find just  $l = 3$  or  $4$ .  $\square$



Although the above presentations are of a special form, we can use this to get general results for Puchta groups.

**Theorem 5.2.** *If  $G$  is a finitely generated group with a Puchta presentation for any prime  $p > 5$  then  $G$  is GS with respect to this presentation.*

*Proof.* We first assume we have a finite presentation

$$\langle x_1, \dots, x_k | w_1^{p^{r_1}}, \dots, w_l^{p^{r_l}} \rangle$$

and we consider the associated GS function

$$f(t) = 1 - kt + \sum_{i=1}^l t^{\deg(w_i^{p^{r_i}})}.$$

But we have  $\deg(w_i^{p^{r_i}}) \geq p^{r_i}$  so  $t^{\deg(w_i^{p^{r_i}})} \leq t^{p^{r_i}}$  for  $t \in (0, 1)$ . Consequently let us define the *Puchta polynomial* (or *Puchta function* in the case of infinite presentations) to be  $g(t) = 1 - kt + \sum t^{p^{r_i}}$ . Then if we can find  $t \in (0, 1)$  with  $g(t) < 0$ , we have  $f(t) < 0$  too. Moreover if any relator is not a proper  $p$ th power (and there are at most  $k - 2$  of these as the  $p$ -deficiency of the presentation is greater than 1) then we can decrease both  $k$  and  $l$  by this amount without changing  $g$ . Thus we may assume that  $k \geq 2$ ; indeed this also holds if the presentation has  $p$ -deficiency exactly 1 unless it has deficiency 1 too. Suppose we now express  $g(t)$  as

$$1 - kt + s_1 t^p + s_2 t^{p^2} + \dots + s_n t^{p^n} \text{ where } 0 \leq s_i \text{ and } 0 < s_n.$$

We define the  $p$ -deficiency of  $g$  to be  $k - \sum_{i=1}^n s_i/p^i$ . Now the idea is to reduce the powers of  $p$  occurring in the terms of  $g$  in an attempt to increase the function, but without changing the  $p$ -deficiency. If  $n = 1$  then we can apply Lemma 5.1. But for  $n > 1$  we can replace  $s_n t^{p^n}$  in  $g(t)$  with the term  $(s_n/p) t^{p^{n-1}}$  to get  $h(t)$  without changing the  $p$ -deficiency of the function. However we have that  $h(t) \geq g(t)$  when  $t^{p^{n-1}} \geq p t^{p^n}$ . For any  $n \geq 2$  we have  $p^{n-1}(p-1) \geq p(p-1)$  so this inequality holds whenever  $1 \geq p t^{p^{n-1}}$ .

We now have a polynomial  $h$  with degree at most  $p^{n-1}$  and we can further turn terms of the form  $t^{p^{n-1}}$  into those of the form  $t^{p^{n-2}}$ , again without changing the  $p$ -deficiency. We can continue doing this until we now have  $g_0(t) = 1 - kt + p r t^p$ , where the  $p$ -deficiency of our presentation is  $k - r > 1$ , with  $g_0(t) \geq g(t)$  on  $[0, p^{-1/(p^2-p)}]$ . Now we can increase  $r$  to  $k - 1$  to finish with  $F(t) = 1 - kt + p(k-1)t^p \geq g_0(t)$  and the  $p$ -deficiency of the function  $F$  is exactly 1. Now Lemma 5.1, in particular the comment in the proof about when the  $p$ -deficiency is equal to 1, tells us that  $F(t)$  has to go below the  $t$ -axis for  $t \in (0, 1)$ . Now we only know that  $f(t) \leq F(t)$  for  $t \leq p^{-1/(p^2-p)} = y_p$ . But evaluating  $F'(t)$  at  $y_p$  gives  $-k + p^2(k-1)y_p^{p-1}$ . Thus  $F'(y_p) > 0$  if  $p^2 p^{-1/p} > k/(k-1)$  which is true for all  $p$  because  $k/(k-1) \leq 2$  and  $p^{1/p} < 2$ . This means that the minimum  $m$  of  $F(t)$ , where  $F'(m) = 0$  and  $F(m) < 0$ , is less than  $y_p$  as  $F'$  is increasing so we have  $f(m) < 0$  as well.

Finally if we have an infinite Puchta presentation  $\langle x_1, \dots, x_k | w_1^{p^{r_1}}, w_2^{p^{r_2}}, \dots \rangle$ , we run through the above proof for each  $n \in \mathbb{N}$  using the presentation defined by taking the first  $n$  relators, from which we obtain the corresponding functions  $f_n(t)$ . From this we see that  $f_n(t) \leq F(t)$  for all  $n$  and for  $t \leq p^{-1/(p^2-p)}$ , thus  $f_n(m) \leq F(m) < 0$  so for our infinite presentation we have that the power series  $f(m)$  converges and is less than 0 too.  $\square$

*Remark.* This proof also works for presentations (finite or infinite) with  $p$ -deficiency equal to 1, with the exception of those finite presentations which also have deficiency 1 as here the Puchta polynomial is  $1 - t$ .

It was shown in [32] that the soluble groups of deficiency 1 are precisely the groups  $G_k \cong \langle a, t | tat^{-1} = a^k \rangle$  for  $k \in \mathbb{Z}$ . In fact these are also the virtually soluble groups. This gives rise to the question: what, in addition to these, are the virtually soluble groups with  $p$ -deficiency equal to 1? By Theorem 5.2 there are no further examples for primes at least 7, because the pro- $p$  completion of a virtually soluble group cannot contain a free pro- $p$  group, as opposed to a GS group. For  $p = 2$  and 3 examples do exist, such as the infinite dihedral group and the  $(3, 3, 3)$  triangle group, but we do not know of an example for  $p = 5$ .

However we must still deal with  $p = 2, 3$  and 5 where the above theorem is not true. We find that even here Puchta groups are very close to being Golod-Shafarevich.

**Theorem 5.3.** *Every Puchta group  $G$  has a Golod-Shafarevich subgroup of finite index, which we can take to be normal and of  $p$ th power index in  $G$ .*

*Proof.* First let us deal with finitely presented Puchta groups. Suppose that

$$G = \langle x_1, \dots, x_d | r_1^{p^{m_1}}, \dots, r_n^{p^{m_n}} \rangle \text{ with Puchta polynomial } f(t) = 1 - dt + \sum_{i=1}^n t^{p^{m_i}}.$$

We use a very similar trick as before, which is that for  $t \in [0, p^{-1/(p-1)}]$  we have  $pt^p \leq t$  and  $pt^{p^k} \leq t^{p^{k-1}}$  for all  $k \geq 1$ . This time we “convert” our terms of the form  $t^{p^{m_i}}$  into linear terms which increases the function on the above range. Suppose that the  $p$ -deficiency of the presentation is  $1 + e$  so the excess  $p$ -deficiency  $e$  is strictly positive. We end up with the function  $g(t) = 1 - (1 + e)t$  which is negative for  $t > 1/(1 + e)$ . Consequently we have that  $G$  is GS if  $p^{-1/(p-1)} > 1/(1 + e)$ .

However  $e$  can be arbitrarily small. But we know  $G$  has a normal subgroup  $H$  of index  $p$ , so on Puchta rewriting we obtain  $p(d - 1) + 1$  generators for  $H$  and the relator  $r_i^{p^{m_i}}$  yields either one relator to the power  $p^{m_i-1}$  or  $p$  relators to the power  $p^{m_i}$ . In particular the contribution of  $r_i^{p^{m_i}}$  to the Puchta polynomial for  $H$  is bounded above by either  $t^{p^{m_i-1}}$  or  $pt^{p^{m_i}}$  and it is the former which is bigger on our range. On again converting our terms to create a linear function, we have an upper bound for this polynomial of the form  $1 - (pe + 1)t$ . Repeating this process  $k$  times which produces a subnormal subgroup  $H_k$  in  $G$  of index  $p^k$ , we have that the Puchta polynomial for  $H_k$ , and hence the GS polynomial, is bounded above by  $1 - (p^k e + 1)t$  on  $[0, p^{-1/(p-1)}]$ . Thus we conclude that  $H_k$  is a GS group once  $p^{-1/(p-1)} > 1/(p^k e + 1)$  and this will happen for large enough  $k$ . Moreover we can assume that  $H_k$  contains  $N \trianglelefteq G$  with index  $p^l$  as mentioned in Section 1, and as  $l \geq k$  the conclusion holds for  $N$  too.

As for the case where  $G$  is infinitely presented with its Golod-Shafarevich function bounded above by  $f(t) = 1 - dt + \sum_{i=1}^{\infty} t^{p^{m_i}}$ , for any  $N \in \mathbb{N}$  let the tail  $T_N(t)$  of  $f$  be  $\sum_{i=N+1}^{\infty} t^{p^{m_i}}$ . Given  $\epsilon > 0$ , we can take  $N$  large enough that  $T_N(t) \leq \epsilon$  for all  $t \in [0, p^{-1/(p-1)}]$ . This is because  $t^{p^k} \leq 1/p^k$  for all  $k \geq 1$  (and for  $k = 0$  too) on the bigger interval  $[0, p^{-1/p}]$  because  $n^{1/n}$  is decreasing in  $n$ . Thus here  $T_N(t) \leq \sum_{i=N+1}^{\infty} 1/p^{m_i}$ , so the right hand side can be taken to be  $\epsilon$ , and this tends to zero as the  $p$ -deficiency is convergent.

If  $G$  has  $p$ -deficiency  $1+e$  as before, we now take  $\epsilon < ep^{-1/(p-1)}/(1-p^{-1/(p-1)})$  so that  $\epsilon/(e+\epsilon) < p^{-1/(p-1)}$ . We have that the GS function for  $G$  is bounded above by

$$f(t) = 1 - dt + \sum_{i=1}^N t^{p^{m_i}} + \sum_{i=N+1}^{\infty} t^{p^{m_i}}.$$

Let  $r = \sum_{i=1}^N 1/p^{m_i}$  so that the  $p$ -deficiency  $1+e = d - r - \epsilon$ . Then if we take a subnormal subgroup  $H_k$  of index  $p^k$  as before and look at the contribution of the first  $N$  relators of  $G$  to the GS function of  $H_k$ , we have as before an upper bound of  $1 - (p^k(d-r-1)+1)t = 1 - (p^k(e+\epsilon)+1)t$  for  $t \leq p^{-1/(p-1)}$ . But in the tail, a relator  $r_i^{p^{m_i}}$  of the given presentation for  $G$  contributes to the GS function of a normal index  $p$  subgroup  $H$  at most either  $t^{p^{m_i-1}}$  or  $pt^{p^{m_i}}$  and both of these are bounded by  $1/p^{m_i-1}$  on our region. Consequently if we define the tail  $T_H$  for  $H$  as those terms arising from the tail of  $G$ , we have  $T_H(t) \leq p\epsilon$  for allowable  $t$ . Hence the contribution of the tail to the GS function of  $H_k$  is at most  $p^k\epsilon$ , giving an upper bound to the whole function of  $p^k\epsilon + 1 - (p^k(e+\epsilon)+1)t$ . This is less than zero for  $t > (p^k\epsilon + 1)/(p^k(e+\epsilon) + 1)$  which tends to  $\epsilon/(e+\epsilon)$  as  $k \rightarrow \infty$ . Thus there are points below the axis in  $[0, p^{-1/(p-1)}]$  for all large  $k$ , so again we can assume that  $H_k$  is normal in  $G$ .  $\square$

*Remark.* In general the GS function of an infinite presentation satisfying the GS inequality need not have radius of convergence 1, for instance  $f(t) = 1 - dt + 4t^2 + 8t^3 + 16t^4 + \dots$  converges only for  $|t| < 1/2$  and this is GS for  $d > 6$ . But an offshoot of the above proof is that this is not true for Puchta groups.

**Corollary 5.4.** *If  $f$  is the GS function of a Puchta presentation with infinitely many relators then  $f$  has radius of convergence 1.*

*Proof.* We do have  $f(1) = \infty$ , so we show convergence when  $t < 1$  and do this for the Puchta function  $g(t) = 1 - dt + \sum_{i=1}^{\infty} t^{p^{m_i}}$  of the presentation. We saw above that  $t^{p^k} \leq 1/p^k$  for all  $k \geq 0$  on  $[0, p^{-1/p}]$  which implies that  $\sum_{i=1}^{\infty} t^{p^{m_i}} \leq \sum_{i=1}^{\infty} 1/p^{m_i}$  which converges. But now given  $l \geq 1$ , take  $N$  such that  $m_i \geq l$  for all  $i \geq N$ . Then for any  $k \geq l$  we have  $t^{p^k} \leq 1/p^k$  on  $[0, p^{-1/p^l}]$  and so  $\sum_{i=N}^{\infty} t^{p^{m_i}} \leq \sum_{i=N}^{\infty} 1/p^{m_i}$ . As  $(p^l)^{-1/p^l}$  tends to 1 as  $l$  tends to infinity, we have convergence on  $[0, 1)$ .  $\square$

The power of these results is that they produce a wide range of Golod-Shafarevich group presentations, both finite and infinite, where the GS condition is confirmed merely by calculating the  $p$ -deficiency rather than having to deal with GS functions. Indeed it is straightforward to produce for any  $p$  a finitely generated  $p$ -group with an infinite presentation which is both Puchta and GS. By Corollary 2.5 this group does not have property (T). Now although GS groups need not have property (T), for instance non abelian free groups are GS, we do not know of any explicit *torsion* GS groups in the literature until now that are proven not to have (T). Recently examples of torsion GS groups with (T) were found and this will be mentioned further in the next section.

We finish with a result on finitely presented Puchta groups.

**Theorem 5.5.** *Every finitely presented Puchta group  $G$  has a strongly Golod-Shafarevich subgroup  $H$  of finite index. In addition,  $H$  can be taken to be of  $p$ th power index and subnormal, or even normal, in  $G$ .*

*Proof.* We prove that there exists  $H \trianglelefteq_{p^N} G$ , for some  $N \in \mathbb{N}$ , satisfying (5.1).

Let us say that  $G$  has a presentation  $\langle X | R \rangle$  with  $d$  generators and  $r$  relators. So  $\text{def}(G; X, R) = d - r$  and we set  $\text{def}_p(G) - 1 = \epsilon > 0$  and  $k$  (which in general will be negative) to be  $d - r - 1$ . As in Section 2, we consider any  $p$ -series

$$\dots \trianglelefteq_p G_n \trianglelefteq_p G_{n-1} \trianglelefteq_p \dots \trianglelefteq_p G_1 \trianglelefteq_p G_0 = G$$

and consider the presentation of  $G_n$  given by successive Puchta rewriting. If we were merely to use the Reidemeister-Schreier procedure, we would have  $\text{def}(G_n) - 1 = p^n(\text{def}(G) - 1)$  but Puchta rewriting always gives a deficiency at least as big, so we have  $\text{def}(G_n) \geq p^n k + 1$  for these Puchta presentations.

From (2.1) we have  $d_p(G_n) \geq \text{def}_p(G_n) = p^n \epsilon + 1$ . When  $p^n \epsilon \geq 2$  this gives us

$$4\text{def}(G_n) + (d_p(G_n))^2 - 4d_p(G_n) = 4\text{def}(G_n) + (d_p(G_n) - 1)^2 - 4 \geq 4 + 4p^n k + (p^n \epsilon - 2)^2 - 4$$

which is quadratic in  $p^n$  with leading term  $\epsilon^2 p^{2n}$ . Thus we have strongly Golod-Shafarevich presentations for all but finitely many  $G_n$ , and from any subnormal  $G_n$  we can continue the series so that  $G_N$  is normal in  $G$  for some  $N \geq n$ .  $\square$

## 6. INFINITE QUOTIENTS WHICH HAVE (T) OR ARE AMENABLE

Amenability is an important and very robust property: it is preserved under subgroups, quotients, extensions and directed unions. Moreover any group containing  $F_2$  cannot be amenable. The first counterexamples to the converse were the torsion groups constructed by Adjan-Novikov and Olshanskii. However the Grigorchuk and Gupta-Sidki examples are amenable.

We can also ask whether there exist residually finite groups not containing  $F_2$  which are non amenable (if one exists then the directed union property means there will be a finitely generated non amenable subgroup which will still be residually finite). This was completely open until [7] by Ershov which showed that there exist GS groups for sufficiently large primes  $p$  with property (T). These will have a  $p$ -quotient which is still GS (hence infinite) and with (T), hence non amenable. Although this does not ensure residual finiteness, the image of such a group in its pro- $p$  completion (which is infinite) is a dense residually finite- $p$  group, thus this image is infinite and has (T), as (T) is preserved by quotients (as well as extensions and finite index subgroups).

Shortly afterwards, other constructions of finitely generated residually finite torsion groups which are non amenable were simultaneously but independently given by Osin in [28] and Schlage-Puchta in [30], and these methods are able to produce  $p$ -groups for all primes  $p$ . In fact once we have existence of such torsion groups, we find many other groups have these as quotients.

**Proposition 6.1.** *Every large finitely generated group  $G$  has a non amenable residually finite torsion quotient, and every  $p$ -large group  $G$  has a non amenable residually finite  $p$ -quotient.*

*Proof.* As in the start of the proof of Corollary 3.6, we can take  $K \trianglelefteq_f G$  (and of  $p$ -power index in the  $p$ -large case) with  $K/L$  equal to such a torsion group (alternatively a  $p$ -group). Now being residually finite and being torsion are preserved under all subgroups, finite index supergroups and finite direct products, therefore by the easier version of Proposition 3.5 mentioned after the proof, we have  $N \trianglelefteq G$  with  $G/N$  being infinite, residually finite and torsion (and if  $K/L$  is a  $p$ -group then so is  $K/N$ , being a subdirect product of  $K/L$ ). As  $G/N$  contains  $K/N$  which surjects onto  $K/L$ , we have that  $G/N$  is non amenable (and if  $K/N$  is a  $p$ -group then so is  $G/K$ , hence  $G/N$  too).  $\square$

In both Osin's and Schlage-Puchta's arguments the method used to show non amenability was that of rank gradient: if  $d(G)$  is the minimum number of generators of the finitely generated group  $G$  then the *rank gradient*  $\text{RG}(G)$  is defined to be the infimum of  $(d(H) - 1)/[G : H]$  over all finite index subgroups  $H$  of  $G$ . If  $\text{RG}(G) > 0$  and  $G$  is finitely presented then it was shown in [20] that  $G$  is non amenable, and this was extended to finitely generated groups in [1].

Schlage-Puchta showed positive rank gradient as follows: on taking a Puchta presentation giving rise to an infinite  $p$ -group  $G$ , any finite index subgroup  $H$  of  $G$  must be subnormal with index a power of  $p$ , because  $H$  must contain  $N \trianglelefteq_f G$  and  $G/N$  is a finite  $p$ -group. Now  $d(H) - 1 \geq d_p(H) - 1 \geq \text{def}_p(H) - 1$  and we have  $(\text{def}_p(H) - 1)/[G : H] = (\text{def}_p(G) - 1)$  which is strictly positive and independent of  $H$ . Although we do not know if  $G$  is necessarily residually finite, the quotient  $G/R_p$ , where  $R_p$  is the intersection of all normal subgroups of  $p$  power index, will be residually finite and will also have positive rank gradient, because any homomorphism from  $H$  to  $(C_p)^n$  must factor through  $R_p$  and  $[G/R_p : H/R_p] = [G : H]$ . We can use this to obtain:

**Proposition 6.2.** *All Puchta groups are non amenable.*

*Proof.* If  $G$  is a group with a presentation  $\langle X | R \rangle$  having  $p$ -deficiency strictly greater than 1 then  $G$  need not be a  $p$ -group, but one can add very high  $p$ th powers of every word in  $F(X)$  whilst still keeping the  $p$ -deficiency higher than 1. Now we have a  $p$ -quotient of  $G$  which must have positive rank gradient and so be non amenable, meaning that  $G$  is too.  $\square$

To return to GS groups, in [28] it is asked whether torsion GS groups have strictly positive rank gradient. By taking a Puchta presentation yielding a  $p$  group and using Theorems 5.2 and 5.3, we see that there must exist GS  $p$ -groups with strictly positive rank gradient for every prime  $p$ .

However there exist GS groups that are not Puchta; indeed if we take Ershov's examples of GS groups with (T) then they cannot even be commensurable with a Puchta group by Corollary 2.5, in contrast to the other way round as proved in the last section. But a major theorem of Ershov in [8], proved a little while after the above results, is that every GS group has an infinite quotient with property (T). This immediately established that all GS groups are non amenable, which was open until then. It also shows that every Puchta group has an infinite quotient with (T) because we know from the last section that any Puchta group either is GS or has a finite index subgroup which is GS, and in the latter case a finite index subgroup having an infinite quotient with (T) implies the whole group does, by Proposition 3.5 (which in this setting is Proposition 4.5 in [8]). So this gives an alternative proof that Puchta groups are non amenable, although a much harder one.

We might ask whether there exist torsion groups which have an infinite quotient with (T) and another infinite quotient that is amenable. They do exist: for instance take an example of each, such as a torsion GS group along with the Grigorchuk or a Gupta-Sidki group (which are themselves infinite amenable  $p$ -groups) and form the direct product. (We can even obtain residually finite examples by making the factors residually finite.) However here we can obtain explicit examples which are both Puchta and GS.

**Corollary 6.3.** *For any prime  $p$  there exists a  $p$ -group which is both Puchta and GS and which has both an infinite amenable quotient and an infinite quotient with (T).*

*Proof.* Take an infinite amenable  $p$ -group, such as the Grigorchuk or a Gupta-Sidki group, and a finite generating set  $X$  for this group. Enumerate the words of  $F(X) - \{\text{id}\}$  as  $w_1, w_2, \dots$  and let

$p^{n_i}$  be the order of  $w_i$  in this group. Now choose  $N_i$  such that  $N_i \geq n_i$  and big enough so that the presentation  $G = \langle X | w_1^{p^{N_1}}, w_2^{p^{N_2}}, \dots \rangle$  is both Puchta and GS.

For instance in the case of the Gupta-Sidki groups when  $p \geq 3$  with the standard generators  $X = \{a, t\}$ , the paper [15] shows that  $n_i$  is at most the word length  $|w_i|$ , so if we order the  $w_i$  in order of word length and then lexicographically, we have  $n_i \leq i$  so  $N_i = i$  will work to make  $G$  Puchta, as well as GS for  $p > 3$  whereas  $N_i = i + 1$  will do for  $p = 3$ . As for the Grigorchuk group when  $p = 2$ , we can take  $X$  to be the standard generating set  $\{a, b, c, d\}$  and  $N_i = i + 3$  will do, as shown in [16] Chapter VIII and associated references.

Thus we have an infinite quotient with (T) by Ershov's result, but  $G$  also surjects onto the relevant Grigorchuk or Gupta-Sidki group because each relation  $w_i^{p^{N_i}} = \text{id}$  holds in this group.  $\square$

We note also that as the Grigorchuk and Gupta-Sidki groups are residually finite,  $G$  has an infinite, residually finite, amenable quotient. This quotient cannot have property  $(\tau)$ , which is implied by (T) although not vice versa, by [22] Example 4.3.3. As  $(\tau)$  is preserved by quotients,  $G$  does not have  $(\tau)$  either.

In [29] a “large” property was defined to be an abstract group property preserved under finite index subgroups and supergroups, as well as prequotients. The standard definition of largeness that we use here was introduced in that paper and shown to be the most restrictive “large” property for the class of finitely generated groups.

Other examples of “large” properties are not having (T) or  $(\tau)$ , not being amenable, but also having an infinite quotient with (T) or  $(\tau)$  or which is amenable, by Theorem 3.5. It seems surprising that the finitely generated  $p$ -groups in Corollary 6.3 with such explicit and straightforward presentations have all six of these “large” properties (as well as there being examples that are residually finite), but because the related Burnside problem is unresolved, we do not know if there exist finitely *presented* torsion groups having the weakest non trivial “large” property: that of being infinite.

## REFERENCES

- [1] M. Abért & N. Nikolov, The rank gradient from a combinatorial viewpoint, arXiv:math/0701925
- [2] S.I. Adjan & P.S. Novikov, Defining relations and the word problem for free periodic groups of odd order. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* 32 (1968) 971-979.
- [3] B. Baumslag & S. J. Pride, Groups with two more generators than relators, *J. London Math. Soc.* 17 (1978) 425-426.
- [4] G. Baumslag, Topics in combinatorial group theory, Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel 1993.
- [5] J. O. Button, Proving finitely presented groups are large by computer, *Experimental Math.* (2010), to appear.
- [6] M. W. Davis, The Geometry and Topology of Coxeter Groups, London Mathematical Society Monographs Series, Vol. 32.
- [7] M. Ershov, Golod-Shafarevich groups with property (T) and Kac-Moody groups, *Duke Math. J.* 145 (2008), no. 2, 309-339.
- [8] M. Ershov, Kazhdan quotients of Golod-Shafarevich groups, arXiv:0908.3734
- [9] E. S. Golod, On nil-algebras and finitely approximable  $p$ -groups. (Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* 28 (1964) 273-276.
- [10] E. S. Golod & I. R. Shafarevich, On the class field tower, *Izv. Akad. Nauk SSSR* 28: (1964) 261-272 (in Russian)
- [11] R. I. Grigorchuk, Degrees of growth of  $p$ -groups and torsion free groups. (Russian) *Mat. sb. (N.S.)* 126(168) (1985), 194-214, 286.

- [12] R. I. Grigorchuk, On Burnside's problem on periodic groups. (Russian) Funktsionalnyi Analiz i ego Prilozheniya, vol. 14 (1980), no. 1, pp. 53-54.
- [13] R. I. Grigorchuk, On torsion images of Coxeter groups and question of Wiegold, arXiv:0912.2758
- [14] R. I. Grigorchuk and P. de la Harpe, Limit behaviour of exponential growth rates for finitely generated groups. Essays on geometry and related topics, Vol. 1, 351–370, Monogr. Enseign. Math., 38, enseignement Math., Geneva, 2001.
- [15] N. Gupta & S. Sidki, On the Burnside problem for periodic groups. Math. Z. 182 (1983), no. 3, 385-388.
- [16] P. de la Harpe, Topics in geometric group theory. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.
- [17] J. Howie, Free subgroups in groups of small deficiency, J. Group Theory 1 (1998) 95-112.
- [18] E. I. Khukhro, Nilpotent groups and their automorphisms, de Gruyter Expositions in Mathematics 8, Walter de Gruyter, Berlin, New York 1993.
- [19] E. I. Khukhro & V. D. Mazurov, Unsolved problems in group theory: the Kourovka notebook, Russian Academy of Sciences, 1995.
- [20] M. Lackenby, Expanders, rank & graph of groups, Israel J. Math. 146 (2005), 357–370.
- [21] M. Lackenby, Detecting large groups, arXiv:0702571
- [22] A. Lubotzky, Discrete groups, expanding graphs and invariant measures. Progress in Mathematics 125, Birkhäuser Verlag, Basel 1994.
- [23] R. C. Lyndon & P. E. Schupp, Combinatorial Group Theory, Berlin: Springer 1977.
- [24] A. Minasyan, A. Yu. Olshanskiĭ, Periodic quotients of hyperbolic and large groups, Gropus, Geom. Dyn. 3 (2009), 423–452.
- [25] B. H. Neumann, Some remarks on infinite groups, J. London Math. Soc. (1) 12 (1937), 120–127.
- [26] P. M. Neumann, The SQ-universality of some finitely presented groups, J. Austral. Math. Soc. **16** (1973) 1–6.
- [27] A. Yu. Olshanskiĭ, Groups of bounded period with subgroups of prime order. (Russian) Algebra i Logika 21 (1982), no. 5, 553–618.
- [28] D. V. Osin, Rank Gradient and torsion groups, arXiv:math/0905.1322
- [29] S. J. Pride, The concept of “largeness” in group theory, in Word problems (II), Stud. Logic Foundations Math. **95**, North-Holland, Amsterdam-New York, 1980, pp. 299–335.
- [30] J.-C. Schlage-Puchta, Profinite Groups Meeting, Imperial College, UK, 12th June 2009.
- [31] Y. Shalom, Rigidity of commensurators and irreducible lattices, Invent. Math. 141 (2000), 1–54.
- [32] J. S. Wilson, Soluble groups of deficiency 1, Bull. London Math. Soc. **28** (1996) 476–480.
- [33] E. Zelmanov, On groups satisfying the Golod-Shafarevich condition. In *New horizons in pro- $p$  groups*, Progr. Math. 184, Birkhäuser Boston, Boston 2000, 223–232.